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## STRING-ORGANIZED FIELD THEORY \*

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### ABSTRACT

A low energy string theory should reduce to an ordinary quantum field theory, but in reality the structures of the two are so different as to make the equivalence obscure. The string formalism is more symmetrical between the spacetime and the internal degrees of freedom, thus resulting in considerable simplification in practical calculations and novel insights in theoretical understandings. We review here how tree or multiloop field-theoretical diagrams can be organized in a string-like manner to take advantage of this computational and conceptual simplicity.

### 1. Introduction

At present energies much less than  $10^{19}$  GeV, excited levels of a superstring cannot be reached, so a superstring scattering amplitude ought not be distinguishable from one obtained from a massless quantum field theory (QFT). Yet the formalism of a string theory is so vastly different from a QFT that this equivalence is not at all obvious. Specifically, for a superstring theory, (1) the fundamental dynamical variables consist of the spacetime  $x^\mu(\sigma, \tau)$  and the internal  $\psi^i(\sigma, \tau)$  fields, all as functions of the worldsheet coordinates  $\sigma$  and  $\tau$ ; (2) these variables propagate as independent free fields throughout the worldsheet, in a manner dependent on the topology but not on the geometry of the worldsheet (reparametrization and conformal invariance); (3) an external photon of momentum  $p$  and wave function  $\epsilon_\mu(p)$  is inserted into the string through a vertex operator  $\epsilon(p) \cdot [\partial_\tau x(\sigma, \tau)] \exp[ip \cdot x(\sigma, \tau)]$ , a form which is fixed by conformal invariance; (4) conformal invariance leads to local (Veneziano) duality<sup>1</sup> of the scattering amplitude, which enables a scattering process to be described by very few string diagrams. In particular, for elastic scattering in the tree approximation, one string diagram (the Veneziano amplitude) gives rise simultaneously to all the  $s$ -channel and the  $u$ -channel exchanges.

In contrast, in QFT, (1') the fundamental dynamical variables are fields  $\psi^i(x)$  of the four-dimensional spacetime coordinates  $x^\mu$ ; (2') these fields propagate freely only between vertices, where interactions take place; (3') external photons are in-

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serted into a Feynman diagram through the photon operator  $\epsilon(p) \cdot A(x) \exp[ip \cdot x]$ ; (4') there are many distinct Feynman diagrams contributing to a scattering amplitude. For elastic scattering,  $s$ - and  $u$ -channel exchanges are given by different diagrams that must be added up together.

The purpose of this talk is to discuss how QFT can be reformulated in a string-like manner. There are two advantages to reformulate QFT this way. On the practical side, it frees the spacetime, spin, and color variables to propagate independently (properties (1) and (2) of string) so that the scattering amplitude can be decomposed into sums of factorized forms much easier to handle. This for example makes it possible for the spinor helicity technique<sup>2</sup> to be applied to loop graphs<sup>3</sup>. The string-like external photon vertex (3) allows gauge invariance to be looked at in a new way<sup>4</sup> that simplifies calculations. If duality (4) can be implemented in QFT as well, then one has effectively a way of summing a number of diagrams. All of these make it possible to compute processes that are difficult or impossible to do in the usual way<sup>5-9</sup>. On the theoretical side, a string-like reformulation of QFT allows new insights to be obtained by studying gauge invariance in gauge and gravitational theories from a new perspective. It might even help the calculation of multiloop string amplitudes by knowing how to do it for the simpler but similarly-formulated multiloop QFT amplitudes.

For that to happen, a proper-time variable  $\tau$  must be introduced into QFT. It is not necessary to introduce the other string variable  $\sigma$  because string excitations are absent so this variable is effectively frozen. Proper time is introduced by the Schwinger representation for scalar propagators, for the Schwinger proper time  $\alpha_r$  for a propagator  $r = (ij)$  joining vertices  $j$  to  $i$  is equal to the proper-time difference  $|\tau_i - \tau_j|$ .

## 2. Spacetime Flow

We shall assume all particles involved to be massless. By making the Schwinger proper-representation for the denominator of every propagator

$$\frac{1}{q^2 + i\epsilon} = -i \int_0^\infty d\alpha \exp(i\alpha q^2) , \quad (2.1)$$

loop-momentum integrations of every QFT amplitude can be carried out. The amplitude for a  $\ell$ -loop Feynman diagram in  $d$ -dimensional space with  $N$  internal lines of momenta  $q_r$  and  $n$  vertices with outgoing momenta  $p_i$  is then<sup>10</sup>

$$A = \int [D\alpha] \Delta(\alpha)^{-d/2} S(q, p) \exp[iP] , \quad (2.2)$$

$$\int [D\alpha] \equiv \left[ \frac{(-i)^{d/2} \mu^\epsilon}{(4\pi)^{d/2}} \right]^\ell i^N \int_0^\infty \left( \prod_{r=1}^N d\alpha_r \right) ,$$

$$P = \sum_{r=1}^N \alpha_r q_r^2 \equiv \sum_{i,j=1}^n Z_{ij}(\alpha) p_i \cdot p_j , \quad (2.3)$$

$$S(q, p) \equiv \sum_{k \geq 0} S_k(q, p) , \quad (2.4)$$

where  $S_k(q, p)$  is obtained from  $S_0(q, p)$  by contracting  $k$  pairs of  $q$ 's in all possible ways, then summing over all the contracted results. The rule for contraction is:

$$q_r^\mu q_s^\nu \rightarrow -\frac{i}{2} H_{rs}(\alpha) g^{\mu\nu} \equiv q_r^\mu \sqcup q_s^\nu . \quad (2.5)$$

These formulas are valid for tree diagrams as well. In that case  $\Delta = 1$  and  $H_{rs} = 0$ .

If the Feynman diagram is regarded as an electric circuit with branch resistances  $\alpha_r$  and outgoing external currents  $p_i$ , then the quantity  $q_r$  in (2.4) and (2.5) becomes the current flowing through the  $r$ th internal line,  $P$  in (2.3) equals to the power consumed in the circuit, and  $Z_{ij}$  is the impedance matrix of the network. The functions  $\Delta$  and  $H_{rs}$  can be attached circuit interpretations as well. In this way all the quantities in the amplitude can be expressed as electric circuit quantities. There are explicit graphical rules<sup>3,10,11</sup> to obtain each of them directly from the Feynman diagram but we will not discuss them here. With the Schwinger parameters  $\alpha_r$  interpreted as proper-time differences, the spacetime flow  $x^\mu(\tau)$  anticipated in the string theory is now given by the flow of currents through the circuit. Color and spin are contained exclusively in the factor  $S(q, p)$ ; their flows will be discussed in the next two sections.

### 3. Color Flow

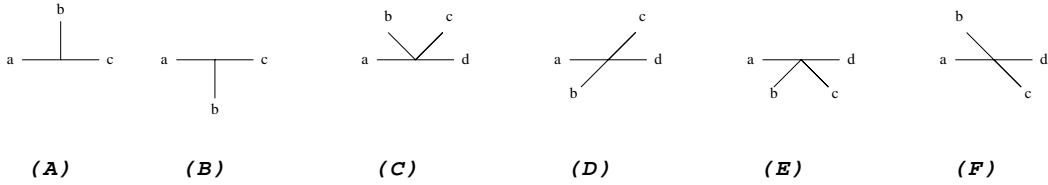
Let  $T^a$  ( $0 \leq a \leq N^2 - 1$ ) be the  $U(N)$  color generators. The color coupling factor for three octet objects is

$$f_{abc} = -i \text{Tr}(T^a T^b T^c - T^a T^c T^b) , \quad (3.1)$$

and that for four octet objects is

$$f_{abe} f_{ecd} = (-i)^2 \text{Tr}(T^a T^b T^c T^d - T^b T^a T^c T^d + T^b T^a T^d T^c - T^a T^b T^d T^c) . \quad (3.2)$$

Instead of a single vertex with a color factor  $f_{abc}$ , one can produce two color-oriented<sup>3,11</sup> (c-o) vertices as in Fig. 1A,B, corresponding to the two terms on the rhs of (3.1): the order in the trace is followed clockwise in the diagrams. Similarly, one has four c-o vertices for the four-gluon coupling as shown in Fig. 1C,D,E,F. These c-o vertices can be obtained from a single one by twisting or flipping some of the lines.



**Fig. 1:** Color-oriented vertices. Read clockwise.

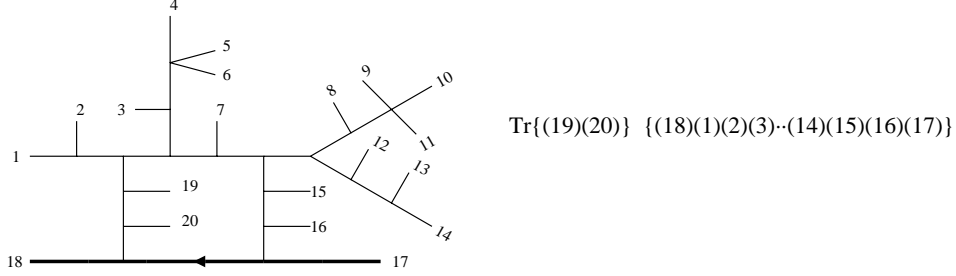
When strung together, they produce c-o diagrams obtainable from a single Feynman diagram by twisting the appropriate lines. So the c-o diagrams are the QFT counterparts of the twisted diagrams in an open string theory. Complete Feynman rules for c-o vertices can be worked out from the usual Feynman rules<sup>3</sup>. Each c-o diagram obtained this way has a fixed color factor, *i.e.*, a fixed color flow, obtained by sewing the  $T^a$ 's together using the  $U(N)$  completeness relation

$$\sum_{a=0}^{N^2-1} (T^a)_{ij} (T^a)_{kl} = \delta_{il} \delta_{kj} . \quad (3.3)$$

The result<sup>2,6</sup> is the generalization of the Chan-Paton factor<sup>12</sup> introduced for string

theory. These color factors can be read off directly from the diagram<sup>3,11</sup> by following the external lines around, with the understanding that every fermion line is to be traversed once on top and every gluon line is to be traversed once on both sides. An open path results in a product of the generators  $T^a$ , and a closed path results in the trace of such products. For example, the color factor from Fig. 2 obtained this way is

$$\text{Tr} (T^{19}T^{20}) T^{18}T^1T^2T^3 \dots T^{14}T^{15}T^{16}T^{17} . \quad (3.4)$$



**Fig. 2:** A color-oriented Feynman diagram. The color factor is given by eq. (3.4)

For  $SU(N)$ , the  $a = 0$  term of (3.3) has to be put on the right and the result is more complicated.

#### 4. Spin Flow

For massless fermions in a gauge theory, helicity and chirality are conserved. This is the reason why fermion helicities can flow freely across the vertex junctions to simulate the free flow (2) found in a string theory. What about photons and gluons, whose helicities are not conserved during interactions? These are spin-1 particles which kinematically can be thought of as composites of two spin- $\frac{1}{2}$  particles. Using this representation chirality conservation can be used to bring about a smooth spin flow across the gauge particles as well.

Mathematically, if  $|p\pm\rangle = u^\pm(p)$ ,  $\langle p\pm| = \bar{u}^\pm(p)$  are the wave functions for massless fermions, then chirality conservation is reflected in the equation  $\langle p_1 \pm | p_2 \pm \rangle = 0$ , leaving only  $\langle p_1 + | p_2 - \rangle \equiv [p_1 p_2]$  and  $\langle p_1 - | p_2 + \rangle \equiv \langle p_1 p_2 \rangle$  non-vanishing<sup>2</sup>. The kinematically-composite nature of the photon and the gluon is mathematically represented by having their wave functions written as  $\epsilon_\mu^\pm(p, k) = \langle p\pm | \gamma_\mu | k\mp \rangle / \sqrt{2} \langle k\mp | p\pm \rangle$ , where  $k$  is an arbitrary massless reference momentum the different choice of which corresponds to a different gauge choice of  $\epsilon$ . The  $\gamma$ -matrices which mix the four spin channels can also be eliminated because of chirality conservation. The relevant formulas are

$$\gamma p_i = |p_i + \rangle \langle p_i + | + |p_i - \rangle \langle p_i - |$$

$$\begin{aligned}
\langle p_1 + |\gamma^\mu| p_2 + \rangle \langle p_3 - |\gamma_\mu| p_4 - \rangle &= 2 \langle p_1 + | p_4 - \rangle \langle p_3 - | p_2 + \rangle \\
\langle p_1 + |\gamma^\mu| p_2 + \rangle \langle p_3 + |\gamma_\mu| p_4 + \rangle &= 2 \langle p_1 + | p_3 - \rangle \langle p_4 - | p_2 + \rangle .
\end{aligned} \tag{4.1}$$

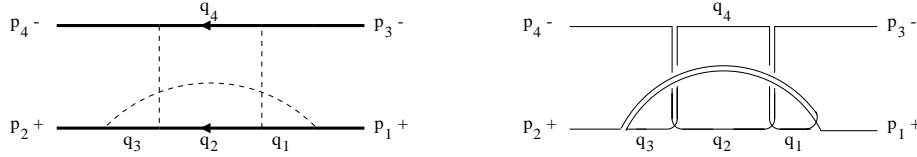
As a result, the spin part of the numerator factor  $S(q, p)$  in (2.2) can be written as sums of products of  $\langle p_i p_j \rangle$  and  $[p_i p_j]$ . A graphical rule can be worked out to read this off directly from the Feynman diagram<sup>3,11</sup>. For that purpose draw each photon and gluon as a pair of quark lines as in Fig. 3, then connect together the quark lines smoothly (according to a rule which we will not state here). Then  $S_0(q, p)$  (and similarly for  $S_k(q, p)$ ) can be obtained by following along the fermion lines, pausing at external lines and internal *fermion* lines to pair up the momenta. Note that along a given fermion line the angular brackets  $\langle \dots \rangle$  and the square brackets  $[\dots]$  always appear alternately. Note also that when offshell momenta appear, they are to be expressed as linear combinations of  $p_i$  (with  $\alpha$ -dependent coefficients) according to the electric-circuit rules of Sec. 2. For example, one can obtain from Fig. 3 in this way

$$S_0 = e^4 \cdot \langle p_4 q_2 \rangle [q_2 p_3] \cdot [p_2 q_1] \langle q_1 q_4 \rangle [q_4 q_3] \langle q_3 p_1 \rangle , \tag{4.2}$$

where

$$\langle p_4 q_2 \rangle [q_2 p_3] \equiv \sum_{j=1}^4 c_{2j}(\alpha) \langle p_4 p_j \rangle [p_j p_3] \tag{4.3}$$

if  $q_r = \sum_{i=1}^4 c_{ri}(\alpha) p_i$ .



**Fig. 3:** A Feynman diagram and its equivalent diagram used to read off the spin factor in  $S_0$ . The result is given in eqs. (4.2) and (4.3).

## 5. String-like Vertex

The string-like vertex (3) can be obtained from the QFT vertex (3') by making use of a series of *differential circuit identities*<sup>4,11</sup>, some of which are shown below:

$$\frac{\partial P}{\partial \alpha_s} = \frac{\partial}{\partial \alpha_s} \left( \sum_r \alpha_r q_r^2 \right) = q_s^2 , \tag{5.1}$$

$$\frac{\partial q_r}{\partial \alpha_s} = H_{rs} q_s , \tag{5.2}$$

$$\frac{\partial H_{rs}}{\partial \alpha_t} = H_{rt} H_{ts} . \tag{5.3}$$

Using these, one can change an external scalar QED vertex,  $V_a = e\epsilon(p_a) \cdot (q_{a'} + q_{a''})$ , where  $p_a$  is the outgoing photon momentum and  $q_{a''}, q_{a'}$  are respectively the charged particle momenta pointing into and out of the vertex, into a string-like form  $V'_a = e\epsilon(p_a) \cdot \partial_{\tau_a} x(\tau_a)$ . As a result, the multiloop amplitude (2.2) for scalar QED

$$A = \int [D\alpha] \Delta(\alpha)^{-d/2} S(q, p) \exp[iP] ,$$

with  $S = \sum_k S_k$ , and

$$S_0 = \left( \prod_{a=1}^{n_A} V_a \right) S_0^{int} , \quad (5.4)$$

$$P = \sum_{ij} p_i \cdot p_j Z_{ij} , \quad (5.5)$$

can be reduced to the string-like formula

$$A = \int [D\alpha] \Delta(\alpha)^{-d/2} S^{int}(q', p) \exp[iP']_{ML} , \quad (5.6)$$

where  $P'$  is obtained from  $P$  by replacing every one of the  $n_A$   $p_a$ 's by  $p_a \rightarrow p_a - ie\epsilon(p_a) \cdot \partial_a$ , where  $\partial_a = \partial/\partial\tau_a = \partial/\partial\alpha_{a''} - \partial/\partial\tau_{a'}$ . The subscript  $ML$  in eq. (5.6) tells us that the exponential should be expanded and only terms multilinear in all  $\epsilon_a$ 's should be kept. Similarly,  $q'_r$  is obtained from  $q_r$  by the same replacement, except that  $p_a \rightarrow p_a - 2ie\epsilon(p_a) \cdot \partial_a$  should be used if  $a$  is one end of  $r$ .

## 6. Local Duality

String theory began with the discovery of the Veneziano model<sup>1</sup> for two-particle scattering. In terms of the Mandelstam variables  $s$  and  $u$ , its amplitude is

$$A(s, u) = -B(-u, -s) = - \int_0^1 x^{-u-1} (1-x)^{-s-1} dx = - \frac{\Gamma(-s)\Gamma(-u)}{\Gamma(-s-u)} . \quad (6.1)$$

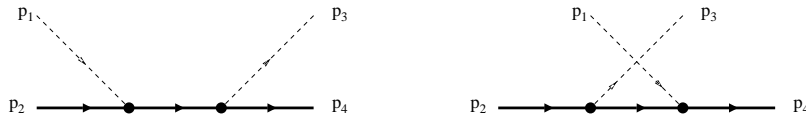
This is a meromorphic function with  $s$ -poles and  $u$ -poles located at non-negative integers (in units of  $[M_P \sim O(10^{19}) \text{ GeV}]^2$ ), but no simultaneous  $s$ - and  $u$ -channel poles. The amplitude can be expanded either as a sum of  $s$ -channel poles, represented purely by  $s$ -channel exchange Feynman diagrams, *or* a sum of  $u$ -channel poles, represented purely by  $u$ -channel exchange diagrams. There is no need to *add* both the  $s$ -channel and the  $u$ -channel diagrams, as is necessary in ordinary quantum field theories. This is *duality* in string theory. What would be a suitable definition of duality in QFT when there is not an infinite tower of massive particles present?

At present energies when  $|s|, |u| \ll 1$  (in units of  $M_P^2$ ), only massless poles in (6.1) contribute, and this yields

$$A(s, u) \simeq \frac{1}{s} + \frac{1}{u} . \quad (6.2)$$

The  $u$ -channel pole comes from the divergence of the integral near  $x = 0$  when  $u = 0$ , and the  $s$ -channel pole comes from the divergence of the integral near  $x = 1$  when  $s = 0$ . In this form, the amplitude does not *appear* to be ‘dual’ because *both* the  $s$ -channel and the  $u$ -channel poles are *summed*, instead of having a *single* expression like (6.1), where only a sum of the  $s$ -channel *or* a sum of the  $u$ -channel poles are present. Nevertheless, appearances are deceiving, because (6.2) follows mathematically from (6.1), which is dual. In other words, the  $u$ -channel poles in (6.2) can be formally obtained by an infinite sum of massive  $s$ -channel poles, and (6.2) is as dual as it can be at low energies.

The amplitude for the ‘Compton scattering’ diagram of Fig. 4, in a massless scalar QFT with interaction  $\phi^* \phi A$ , is just given by (6.2) and therefore already dual. The only difference one can point to is that  $A$  in (6.1) is given by a *single* integral, whereas (6.2) contains a sum of two terms. We shall therefore define a *dual amplitude* in QFT as a sum of Feynman diagrams which can be expressed as a single integral. The question is whether QFT Feynman diagrams can be summed up to give dual amplitudes.



**Fig. 4:** The  $s$ -channel and the  $u$ -channel diagrams for Compton scattering.

The answer is yes for QED-like theories<sup>13</sup> but it is not yet clear for QCD. What makes QED special is that the different diagrams in a gauge-invariant sum of diagrams can be distinguished by the orderings of their photon vertices along the charged lines. If each vertex is assigned a proper time  $\tau$ , then using (2.2) and expressing the  $\alpha$ 's as differences of the  $\tau$ 's, every diagram is given by an integral over the  $\tau$ 's. The integration region for different diagrams are different and non-overlapping. Symbolically, one can write the amplitude (2.2) of a diagram to be

$$A = \int_R [D\tau] T(\tau, p) \exp[iP(\tau, p)] , \quad (6.3)$$

where the integration region  $R$  differs from diagram to diagram. Since the different  $R$ 's do not overlap, one can define an overall function  $T$  and an overall function  $P$  in the union  $C$  of the  $R$ 's (which turns out to be a hypercube) to be equal to



the respective values of  $T$  and  $P$  in specific  $R$  regions. In this way, the dual sum of these gauge-invariant diagrams is simply

$$A_{sum} = \int_C [D\tau] T(\tau, p) \exp[iP(\tau, p)] . \quad (6.4)$$

How useful such a formal sum is depends on our ability to evaluate the integral over the hypercube  $C$ . In general this is not easy but in the eikonal approximation this can be done<sup>13</sup>. Even when (6.4) cannot be evaluated exactly it is still useful for two reasons. It can be the starting point of an approximation (such as the eikonal approximation) to sum diagrams. It can also be used for mathematical manipulations (*e.g.*, the integration-by-parts technique<sup>6,8,9</sup>), effectively to shift gauge-dependent contributions between different diagrams so that they do not appear even in the intermediate steps.

## 6. Summary

A string-theory scattering amplitude at an energy much below the Planck mass should be the same as an appropriate massless QFT amplitude, but on the surface they look different until one uses the Schwinger-parameter representation. When this is done, spacetime, spin, and color flows can be separated, spinor helicity technique and string-like gauge vertices can be used to simplify the calculations and to obtain novel insights into the structure of gauge and gravitational theories. String-like local duality can also be implemented in QED-like theories but its feasibility is not yet clear for QCD.

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